Exercise 3
Prove that the iteration in the Babylonian method above converges quadratically to the square root of x. In particular, show that the error $\epsilon_n = \frac{x_n}{\sqrt{x}} - 1$ satisfies

$$\epsilon_{n+1} = \frac{\epsilon_n^2}{2(\epsilon_n + 1)}.$$

From this, we get that $\epsilon_n \geq 0$ for $n \geq 1$, and so

$$\epsilon_{n+1} \leq \frac{1}{2}\min\{\epsilon_n^2, \epsilon_n\}.$$

Why does the inequality above guarantee convergence (i.e., that $\epsilon_n \to 0$ as $n \to \infty$)?

Proof: We first show that

$$\epsilon_{n+1} = \frac{\epsilon_n^2}{2(\epsilon_n + 1)}.$$

To this end, we first right down the right hand side of the equation:

$$\begin{aligned}
\frac{\epsilon_n^2}{2(\epsilon_n + 1)} &= \frac{(\frac{x_n}{\sqrt{x}} - 1)^2}{2(\frac{x_n}{\sqrt{x}} - 1 + 1)} \\
&= \frac{(\frac{x_n}{\sqrt{x}} - 1)^2}{\frac{2x_n}{\sqrt{x}}} \\
&= \frac{x_n}{2\sqrt{x}} - 1 + \frac{\sqrt{x}}{2x_n}
\end{aligned}$$

Moreover, we notice that

$$\begin{aligned}
\epsilon_{n+1} &= \frac{x_{n+1}}{\sqrt{x}} - 1 \\
&= \frac{\frac{1}{2}(x_n + \frac{x}{x_n})}{\sqrt{x}} - 1 \\
&= \frac{\frac{1}{2}x_n + \frac{x}{2x_n}}{\sqrt{x}} - 1 \\
&= \frac{x_n}{2\sqrt{x}} + \frac{\sqrt{x}}{2x_n} - 1.
\end{aligned}$$

Therefore, we indeed have the right hand side to be equal to the left hand side. Note that we have $x_0 = x > 0$ by definition (From Exercise 2). Therefore, we must have $\epsilon_n \geq 0$ for all $n \geq 1$. Moreover, we note that

$$\epsilon_{n+1} \leq \frac{1}{2}\min\{\epsilon_n^2, \epsilon_n\}.$$

This inequality can be derived from a simple case discussion. Now we shall prove that the tolerance term $\epsilon_n \to 0$ as $n \to \infty$. To this end, we note that we have $\epsilon_n \geq 0$ for all $n \geq 1$. Moreover, we have $\epsilon_{n+1} \leq \frac{\epsilon_n}{2} \leq \epsilon_n$ for all n. In particular, this means that we have a decreasing sequence $\{\epsilon_n\}_{n=1}^{\infty}$. Therefore, we must have the sequence to converge by the monotone convergence theorem. Now we ask the question, where the sequence converges to. To this end, we let $\epsilon = \lim_{n\to\infty} \epsilon_n$. Therefore, we must have

$$\begin{aligned}
\epsilon &= \frac{\epsilon^2}{2(\epsilon + 1)} \\
2\epsilon^2 + 2\epsilon &= \epsilon^2 \\
\epsilon^2 + 2\epsilon &= 0 \\
\epsilon(\epsilon + 2) &= 0.
\end{aligned}$$

Therefore, we must have $\epsilon = 0$ or $\epsilon = -2$. However, we have $\epsilon \geq 0$. Therefore, we must have

$$\epsilon = \lim_{n \to \infty} \epsilon_n = 0. \quad \square$$